Lemma 7.13 $(L,G,\gamma)\cong(L,G,\gamma')$ as K-algebras

$$\Leftrightarrow \exists \mu: G\to L^\times: \quad \gamma'(\sigma,\tau)=\frac{\mu(\sigma)\,\sigma(\mu(\tau))}{\mu(\sigma\tau)}\,\gamma(\sigma,\tau)$$

Proof: "$\Rightarrow$": $R\cong(L,G,\gamma)$, $R'\cong(L,G,\gamma')$, $f: R\xrightarrow{\sim} R'$

$j_1: L\hookrightarrow R$, $j_2: L\hookrightarrow R'$

$\overset{T6.10}{\Longrightarrow} \exists u\in(R')^\times$: $\underbrace{f\circ j_1(\lambda)=u\,j_2(\lambda)\,u^{-1}}$ w.l.o.g. $u=1$, so $f|_L=\mathrm{id}_L$

Let $(x_\sigma)_{\sigma\in G}$ be an L-basis of R s.t. $x_\sigma x_\tau=\gamma(\sigma,\tau)x_{\sigma\tau}$, $x_\sigma\lambda=\sigma(\lambda)x_\sigma$

$(x'_\sigma)_{\sigma\in G}$ — " — R' s.t. $x'_\sigma x'_\tau=\gamma'(\sigma,\tau)\,x'_{\sigma\tau}$, $x'_\sigma\lambda=\sigma(\lambda)x'_\sigma$ $(\lambda\in L)$

Now $(f(x_\sigma))_{\sigma\in G}$ is an L-basis of R' with

$$f(x_\sigma)f(x_\tau)=\underbrace{\gamma(\sigma,\tau)}_{\in L}f(x_{\sigma\tau})$$

Also: $\forall\lambda\in L$: $f(x_\sigma)\lambda f(x_\sigma)^{-1}=f(x_\sigma\lambda x_\sigma^{-1})=f(\sigma(\lambda))=\sigma(\lambda)$

$\overset{L7.9}{\Longrightarrow}\exists\mu: G\to L^\times$: $x'_\sigma=\mu(\sigma)f(x_\sigma)$

$$\Rightarrow \mu(\sigma)f(x_\sigma)\mu(\tau)f(x_\tau)=\gamma'(\sigma,\tau)\mu(\sigma\tau)f(x_{\sigma\tau})$$

$$\mu(\sigma)\,\sigma(\mu(\tau))\,f(x_\sigma x_\tau)=\mu(\sigma)\sigma(\mu(\tau))\,\gamma(\sigma,\tau)\,f(x_{\sigma\tau})$$

$$\Rightarrow \gamma'(\sigma,\tau)=\frac{\mu(\sigma)\,\sigma(\mu(\tau))}{\mu(\sigma\tau)}\gamma(\sigma,\tau)$$

"$\Leftarrow$" Define $f: (L,G,\gamma')\longrightarrow(L,G,\gamma)$, $\quad$ (L-v.s. isomorphism)

$$x'_\sigma\longmapsto\mu(\sigma)x_\sigma$$

Then: $f(x'_\sigma x'_\tau)=f(\gamma'(\sigma,\tau)x'_{\sigma\tau})=\gamma'(\sigma,\tau)\mu(\sigma\tau)\,x_{\sigma\tau}$

$f(x'_\sigma)f(x'_\tau)=\mu(\sigma)x_\sigma\,\mu(\tau)x_\tau=\mu(\sigma)\,\sigma(\mu(\tau))\,x_\sigma x_\tau$

$$= \mu(\sigma)\,\sigma(\mu(\tau))\,\gamma(\sigma,\tau)\,x_{\sigma\tau} = \gamma'(\sigma,\tau)\,\mu(\sigma\tau)\,x_{\sigma\tau}. \qquad \square$$

L/K finite Galois ext., $G = Gal(L/K)$

If γ, γ' are 2-cocycles,

$$\gamma' \sim \gamma :\Leftrightarrow \exists \mu : G \to L^\times : \quad \gamma'(\sigma,\tau) = \frac{\mu(\sigma)\,\sigma(\mu(\tau))}{\mu(\sigma\tau)}\,\gamma(\sigma,\tau) \quad (\sigma,\tau \in G)$$

<u>Thm 7.14</u> There is a bijection

$$Br(L/K) \leftrightarrow \{2\text{-cocycles } \gamma : G \times G \to L^\times\}/_\sim$$

<u>Proof:</u> „$\mapsto$" Every class in $Br(L/K)$ has a unique (up to iso) repr. R containing L as strict max. subfield [T7.8]

$\overset{\text{Sec 7}}{\Longrightarrow} R \cong (L, G, \gamma)$ for some γ

By L7.13, $[R] \mapsto [\gamma]_\sim$ is well-defined and injective.

By T7.12, the map is surjective. $\square$

7.4 Homological characterization of $Br(L/K)$

Group cohomology: G group, $(M,+)$ abelian group on which G acts

($G \times M \to M$, $(g,m) \mapsto gm$ s.t. $\forall m,n \in M, \forall g,h \in G$:

$$1_G m = m, \qquad g(hm) = (gh)m, \qquad g(m+n) = gm + gn$$

$\Leftrightarrow$ M is a $\mathbb{Z}[G]$-module)

$\forall n \geq 0$: $C^n(G,M) := \{\text{maps } f : G^n \to M\}$ $\quad$ ($G^0 = \{1_G\}$, so $C^0(G,M) = M$)

elements are n-cochains of G with values in M.

$C^n(G,M)$ is an abelian group w. pointwise addition

$C^n(G,M)$ ... n-th cochain group

Define $\delta^n : C^n(G,M) \longrightarrow C^{n+1}(G,M)$ by

$$(\delta^n f)(g_1,\dots,g_{n+1}) = g_1 f(g_2,\dots,g_{n+1}) + \sum_{i=1}^{n} (-1)^i f(g_1,\dots,g_{i-1},\underbrace{g_i g_{i+1}}_{i\text{-th place}},g_{i+2},\dots,g_{n+1})$$
$$+ (-1)^{n+1} f(g_1,\dots,g_n)$$

Exm: $(\delta^0 \underset{\substack{\cap \\ M}}{f})(g_1) = g_1 \cdot f - f$

$$(\delta^1 f)(g_1,g_2) = g_1 f(g_2) - f(g_1 g_2) + f(g_1)$$

$$(\delta^2 f)(g_1,g_2,g_3) = g_1 f(g_2,g_3) - f(g_1 g_2, g_3) + f(g_1, g_2 g_3) - f(g_1,g_2)$$

δ^n ... n-th coboundary map (group hom.)

Lemma 7.15: $\delta^{n+1} \circ \delta^n = 0 \quad \forall n \geq 0$.

Proof: Substitute & check everything cancels. We only do $n=0$, $n=1$:

$$(\delta^1 \circ \delta^0 \underset{\substack{\cap \\ M}}{f})(g_1,g_2) = g_1 (\delta^0 f)(g_2) - (\delta^0 f)(g_1 g_2) + (\delta^0 f)(g_1)$$
$$= g_1(\cancel{g_2 f} - \cancel{f}) - (\cancel{g_1 g_2 f} - \cancel{f}) + (\cancel{g_1 f} - \cancel{f}) = 0.$$

$$(\delta^2 \circ \delta^1 f)(g_1,g_2,g_3) = g_1 (\delta^1 f)(g_2,g_3) - (\delta^1 f)(g_1 g_2, g_3) +$$
$$(\delta^1 f)(g_1, g_2 g_3) - (\delta^1 f)(g_1,g_2)$$
$$= g_1 (\cancel{g_2 f(g_3)} - \cancel{f(g_2 g_3)} + \cancel{f(g_2)})$$
$$- \cancel{g_1 g_2 f(g_3)} + \cancel{f(g_1 g_2 g_3)} - \cancel{f(g_1 g_2)}$$
$$+ \cancel{g_1 f(g_2 g_3)} - \cancel{f(g_1 g_2 g_3)} + \cancel{f(g_1)}$$
$$- \cancel{g_1 f(g_2)} + \cancel{f(g_1 g_2)} - \cancel{f(g_1)} = 0$$

$\square$

$$0 \xrightarrow{\delta^{-1}} C^0(G,M) \xrightarrow{\delta^0} C^1(G,M) \xrightarrow{\delta^1} C^2(G,M) \to \dots$$

is a cochain complex.

$Z^n(G,M) := \ker(\delta_n)$ n-cocycles

$B^n(G,M) := \operatorname{im}(\delta_{n-1})$ n-coboundaries

$\delta^{n+1} \circ \delta^n = 0 \to B^n \leq Z^n \xRightarrow{Z^n \text{ abelian}} Z^n/B^n$ abelian group

Def: The n-th cohomology group of G w. coefficients in M is

$$H^n(G,M) := Z^n(G,M)/B^n(G,M).$$

Exm: $H^0(G,M) = \overbrace{\{m \in M : \forall g \in G : gm = m\}}^{\ker \delta^0}$ is the set of G-invariant elements of M.

Def: L/K Galois extension, $G := \mathrm{Gal}(L/K)$

$H^n(G, L^\times)$ is the n-th Galois cohomology group of L/K w. coeffs in $L^\times$.

Here: $H^0(G, L^\times) = K^\times$ (K is the fixed field of G in L)

$H^1(G, L^\times) = \{1\}$ (Noether, " Hilbert's Thm 90 ") $\to$ later

$\gamma \in Z^2(G, L^\times) \iff \delta^2(\gamma) = 1$

$$1 = \delta^2(\gamma)(\underbrace{\rho, \sigma, \tau}_{\in G}) = \rho(\gamma(\sigma,\tau))\, \gamma(\rho\sigma,\tau)^{-1}\, \gamma(\rho,\sigma\tau)\, \gamma(\rho,\sigma)^{-1}$$

$$\iff \boxed{\rho(\gamma(\sigma,\tau))\, \gamma(\rho,\sigma\tau) = \gamma(\rho\sigma,\tau)\, \gamma(\rho,\sigma)}$$

$\iff \gamma$ 2-cocycle in sense of Sec. 7.3.

$B^2(G,L^\times)$: $\operatorname{im}\delta^1$

For $\mu: G\to L^\times$, $\delta^1(\mu)(\underset{G}{\sigma},\underset{G}{\tau}) = \sigma(\mu(\tau))\cdot\mu(\sigma\tau)^{-1}\cdot\mu(\sigma)$

So, for $\gamma,\gamma'\in Z^2(G,L^\times)$: $\gamma\sim\gamma' \Leftrightarrow \gamma'\gamma^{-1}\in B^2(G,L^\times)$
as in L7.13

__Cor 7.16__ If L/K is Galois, $G=\mathrm{Gal}(L/K)$, then is a bijection of sets $H^2(G,L^\times)\longrightarrow \mathrm{Br}(L/K)$, $[\gamma]\longmapsto[(L,G,\gamma)]$ crossed product.

__Proof:__ T7.14 + preceding discussion. $\square$

__Thm 7.17:__ L/K Galois, $G=\mathrm{Gal}(L/K)$. Then the map from C7.16 is a group isomorphism.

__Proof:__ Need to check: if α,β are 2-cocycles, then

$$[(L,G,\alpha)]+[(L,G,\beta)]=[(L,G,\alpha\beta)] \text{ in } \mathrm{Br}(L/K)$$

Let $A:=(L,G,\alpha)$, $B:=(L,G,\beta)$, $C:=(L,G,\alpha\beta)$

__Show:__ $A\otimes_K B\sim C$ (as K-algs)

$M:=A^{op}\otimes_L B$, (so $\forall\lambda\in L$: $(\lambda a)\otimes b=(a\cdot_{op}\lambda)\otimes b=a\otimes\lambda b$)

__right $A\otimes_K B$-module structure on M:__

$$\underbrace{(a'\otimes_L b')}_{\in M}\cdot(a\otimes_K b):=a'a\otimes_L b'b$$

[well-definedness uses A^{op}; e.g. $(\lambda a'\otimes_L b')(a\otimes_K b)=(\lambda a'a\otimes_L b'b)=a'a\otimes_L\lambda b'b$
$=(a'\otimes_L\lambda b')(a\otimes_K b)$]

<u>left C-module structure on M:</u>

Let $(z_\sigma)_{\sigma\in G}$ an L-basis of C s.t. $z_\sigma z_\tau = (\alpha\beta)(\sigma,\tau)\, z_{\sigma\tau}$, $z_\sigma \lambda = \sigma(\lambda) z_\sigma$

analogously $(x_\sigma)_{\sigma\in G}$, $(y_\sigma)_{\sigma\in G}$ L-basis for A & B

Define $(\lambda z_\sigma)(a\otimes_L b) := (\lambda x_\sigma a)\otimes y_\sigma b$ (check well-definedness!)

This is a left C-module structure on M:

We only check: $(cc')m = c(c'm)$ $\forall m\in M$, $c,c'\in C$

Let $m = a\otimes b$, $c = \lambda z_\sigma$ $c' = \lambda' z_\tau$

$$(\lambda z_\sigma \lambda' z_\tau)(a\otimes_L b) = (\lambda \sigma(\lambda')\cdot(\alpha\beta)(\sigma,\tau)\, z_{\sigma\tau})(a\otimes_L b)$$

$$= (\overbrace{\lambda}^{\in L}\overbrace{\sigma(\lambda')}^{\in L}\overbrace{\alpha(\sigma,\tau)}^{\in L}\cdot\overbrace{\beta(\sigma,\tau)}^{\in L}\, x_{\sigma\tau} a)\otimes y_{\sigma\tau} b$$

$$= (\lambda\,(\sigma(\lambda')\alpha(\sigma,\tau)\, x_{\sigma\tau}\, a))\otimes(\beta(\sigma,\tau)\, y_{\sigma\tau}\, b)$$

$$= (\lambda x_\sigma \underline{\lambda' x_\tau a})\otimes(y_\sigma \underline{y_\tau b}) = \lambda z_\sigma(\lambda' z_{\tau}(a\otimes_L b)).$$

Additionally M is a $(C, A\otimes_K B)$-bimodule

$$\Longrightarrow \exists\ K\text{-algebra hom } \varphi: \begin{cases} (A\otimes_K B)^{op} \longrightarrow \mathrm{End}({}_C M) \\ a\otimes b \longmapsto (x\mapsto x(a\otimes b)) \end{cases}$$

<u>Show</u>: φ is an isomorphism.

φ injective, bec $A\otimes_K B$ is simple.

$$n := [L:K] \Longrightarrow \dim_L A = \dim_L B = n \Longrightarrow \dim_L M = n^2 \overset{[L:K]=n}{\Longrightarrow} \dim_K M = n^3.$$

$\dim_K C = n^2$

$$C \text{ f.d. simple} \overset{L6.12}{\Longrightarrow} M \cong_C C^n$$

$$\Rightarrow End({}_C M) \cong End({}_C C^n) \cong M_n(End({}_C C)) \cong M_n(C^{op}) \cong C^{op} \otimes_K M_n(K)$$

$$\Rightarrow \dim_K(End_C M) = n^2 \dim_K C = n^4 = \dim_K(A \otimes_K B)$$

$\Rightarrow \varphi$ is an isomorphism and $End({}_C M) \sim C^{op}$.

$$\Rightarrow (A \otimes_K B)^{op} \sim C^{op} \Rightarrow C \sim A \otimes_K B \qquad \square$$

Thm 7.18 If $|G| < \infty$, then $|G| \cdot H^n(G, M) = 0 \quad \forall n, M$

Proof: Let $f \in Z^n(G, M)$. Show: $|G| \cdot f \in B^n(G, M) = \operatorname{im} \delta^{n-1}$

$$0 = (\delta^n f)(g_1, \dots, g_{n+1}) = g_1 f(g_2, \dots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i f(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1})$$

$$+ (-1)^{n+1} f(g_1, \dots, g_n)$$

$$\Rightarrow (-1)^n f(g_1, \dots, g_n) = g_1 f(g_2, \dots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i f(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1})$$

sum over $g_{n+1} \in G$

$$\Rightarrow (-1)^n |G| f(g_1, \dots, g_n) = \sum_{g_{n+1} \in G} \left(g_1 f(g_2, \dots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i f(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}) \right)$$

$$h(\tilde g_2, \dots, \tilde g_n) := \sum_{g_{n+1} \in G} f(\tilde g_2, \dots, \tilde g_n, g_{n+1}) \in C^{n-1}(G, M)$$

$$\Rightarrow (-1)^n |G| f(g_1, \dots, g_n) = g_1 h(g_2, \dots, g_n) + \sum_{i=1}^{n-1} (-1)^i h(g_1, \dots, g_i g_{i+1}, \dots, g_n)$$

$$+ (-1)^n h(g_1, \dots, g_{n-1}) = \delta^{n-1}(h). \qquad \square$$

Thm 7.19 For any field K, $Br(K)$ is a torsion abelian group.

Proof: $Br(K) \overset{[T7.7]}{=} \bigcup (Br(L/K) : L/K \text{ Galois})$, and $Br(L/K)$ is torsion by [T7.17], [T7.18].

$\square$